EL1010 - SESSION 1 RUSSO ALESSIO

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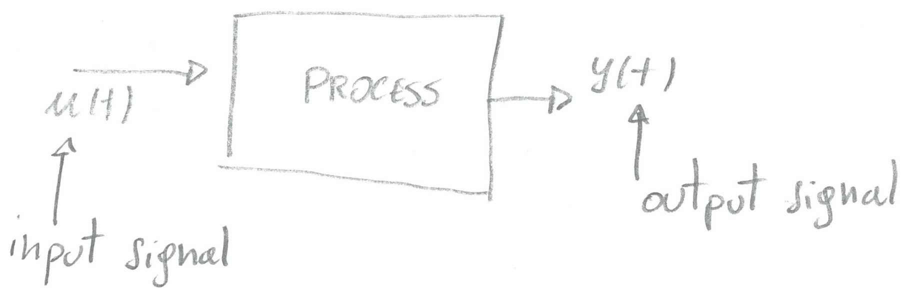
· **Dynamical Systems**: Processes that change with time.

We will focus on processes that can be modelled using differential equations, like

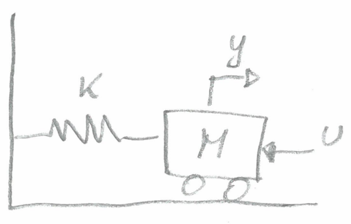
$$y(t) + \dot{y}(t) = u(t)$$

where $\dot{y}(t) = \frac{d}{dt} y(t)$, and y is the quantity to find. We can represent this using INPUT/OUTPUT blocks:



Thanks to this notation we can understand the direction of causality, i.e., that $u(t)$ causes $y(t)$, and not the opposite!!

EXAMPLE



Consider a simple "MASS-SPRING" SYSTEM, without friction. The cart has mass M and the spring a stiffness constant K. NEWTON'S 2ND LAW tells us that

$M\ddot{y} = -Ky$. Now add an external force u, as in figure, then:

$$M\ddot{y} = -Ky + u \Rightarrow \frac{d^2 y(t)}{dt^2} + \frac{K}{M} y(t) = u(t) \cdot \frac{1}{M}$$

N. B. (NOTA BENE): Initial conditions are important!

LAPLACE + TRANSFER FUNCTIONS

Definition: Laplace Transform

Given a function $f(t): \mathbb{R} \to \mathbb{R}$, defined for $t \geq 0$, the LAPLACE TRANSFORM of f is given by $F(s)$:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{+\infty} f(t) e^{-st} dt, \quad s \in \mathbb{C}, \quad s = \sigma + j\omega$$

Why do we use the Laplace Transform?

1) Similarly to the Fourier Transform, the Laplace Transform allows us to study the differential equation in the frequency domain, and not in the time domain.

2) Solving a differential equation using the Laplace transform is usually easier, because you work with polynomials, and not derivatives!

→ To understand this, check the following property:

TIME DERIVATIVE RULE

The Laplace transform of $\dot{f}(t) = \frac{d}{dt} f(t)$ is:

$$\mathcal{L}\{\dot{f}(t)\} = \int_0^{+\infty} \dot{f}(t) e^{-st} dt \underset{\text{integration by parts}}{=} f(t) e^{-st} \Big|_0^{+\infty} + s \int_0^{+\infty} f(t) e^{-st} dt$$

$$= \left(\lim_{t \to \infty} f(t) e^{-st} - f(0) e^{0} \right) + s F(s) = sF(s) - f(0)$$

EXAMPLE

We can try to apply the previous rule to $\ddot{y}(t) + \frac{K}{M} y(t) = \frac{1}{M} u(t)$.
Denote by $Y(s) = \mathcal{L}\{y(t)\}$ the Laplace transform of y, and by $U(s) = \mathcal{L}\{u(t)\}$ the Laplace transform of u.

$$\mathcal{L}\{\ddot{y}(t) + \frac{K}{M} y(t)\} = \mathcal{L}\{\frac{1}{M} u(t)\}$$
$$\downarrow$$
$$\mathcal{L}\{\ddot{y}(t)\} + \frac{K}{M}\mathcal{L}\{y(t)\} = \frac{1}{M}\mathcal{L}\{u(t)\}$$
$$\downarrow$$
$$\mathcal{L}\{\ddot{y}(t)\} + \frac{K}{M} Y(s) = \frac{1}{M} U(s)$$

NB. the Laplace Transform is a LINEAR OPERATOR!
$\Rightarrow \mathcal{L}\{a x(t) + b y(t)\} = a\mathcal{L}\{x(t)\} + b\mathcal{L}\{y(t)\}$

What is $\mathcal{L}\{\ddot{y}\}$? Denote $x = \dot{y}$, then $\dot{x} = \ddot{y}$

$\Rightarrow \mathcal{L}\{\ddot{y}\} = \mathcal{L}\{\dot{x}\} = sX(s) - x(0)$, where $X(s) = \mathcal{L}\{x(t)\}$.

I used the time derivative rule

Now, notice that $X(s) = \mathcal{L}\{x(t)\} = \mathcal{L}\{\dot{y}(t)\} = sY(s) - y(0)$

$$\Rightarrow \mathcal{L}\{\ddot{y}\} = sX(s) - \underset{\dot{y}(0)}{\underset{\|}{x(0)}} = s(sY(s) - y(0)) - \dot{y}(0) = s^2Y(s) - sy(0) - \dot{y}(0)$$

The final equation is

$$s^2Y(s) - sy(0) - \dot{y}(0) + \frac{K}{M}Y(s) = \frac{1}{M}U(s)$$

Suppose $y(0) = \dot{y}(0) = 0$, then $\rightarrow s^2Y(s) + K/M\, Y(s) = 1/M\, U(s)$

Then $Y(s)(s^2 + K/M) = 1/M\, U(s)$

$$\Rightarrow Y(s) = \underbrace{\frac{1}{M}\frac{1}{s^2 + K/M}} U(s)$$

(ÖVERFÖRINGSFUNKTION)
THIS IS ALSO CALLED TRANSFER FUNCTION BETWEEN U AND Y! (we will call it G(s) or T(s))

The solutions of $s^2 + K/M = 0$ are called POLES of the system.

· DEFINITION (POLES) (POLER)

The roots of the DENOMINATOR of the transfer function are called POLES, or EIGENVALUES of the associated differential equation

· DEFINITION (ZEROS) (nollställen)

The roots of the NUMERATOR of the transfer function are called ZEROS.

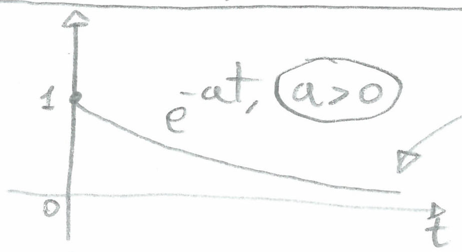
· Poles are related to the STABILITY (STABILITET) of a PROCESS! Why?

Suppose the process is modelled by $\dot{y} + ay = u$, the transfer function is

$$sY(s) - y(0) + aY(s) = U(s) \qquad (\text{suppose } y(0) = 0)$$

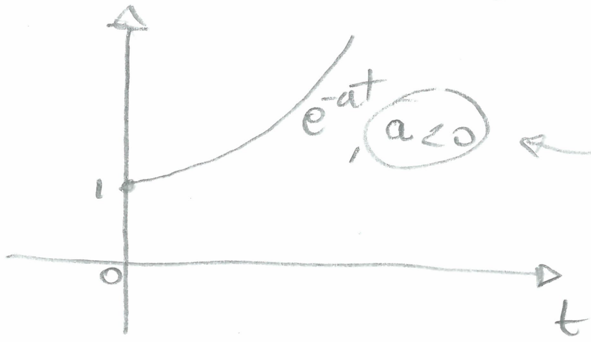
$$Y(s) = \frac{1}{s+a} U(s) \Rightarrow T(s) = \frac{1}{s+a} \text{ is the transfer function}$$

The pole is given by the solution of $s + a = 0 \Rightarrow s = -a$. If $a > 0$ the system is STABLE! The inverse Laplace transform of $T(s)$ is e^{-at}:



goes to 0 for $t \to \infty$!

$\Rightarrow$ STABLE!



goes to ∞ for $t \to \infty$

$\Rightarrow$ UNSTABLE

- **ASYMPTOTIC STABILITY**: The system is asymptotically stable if all the poles have strictly negative real part
- **UNSTABLE SYSTEM**: The system is unstable if there is at least 1 pole with strictly positive real part

EX: **ASYMPTOTICALLY STABLE**

$$T(s) = \frac{1}{s^2+2s+1} \rightarrow s^2+2s+1=0 \rightarrow \begin{cases} s_1 = -1 < 0 \\ s_2 = -1 < 0 \end{cases}$$

EX. **UNSTABLE**

$$T(s) = \frac{1}{s^2-1} \rightarrow s^2-1=0 \rightarrow \begin{cases} s_1 = 1 > 0 \leftarrow \text{UNSTABLE POLE} \\ s_2 = -1 < 0 \end{cases}$$

- **What about ZEROS?**

Zeros do not affect stability, but performance. A negative zero may cause OVERSHOOT (a positive zero may cause UNDERSHOOT). ZEROS may also speed up the response of the system

THEOREM: FINAL VALUE THEOREM (SLUTVÄRDESSATSEN)

If all the poles of $T(s)$ have strictly negative real part, then

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sT(s)U(s) = \lim_{s\to 0} sY(s)$$

EXAMPLE: STEP RESPONSE (STEGSVAR)

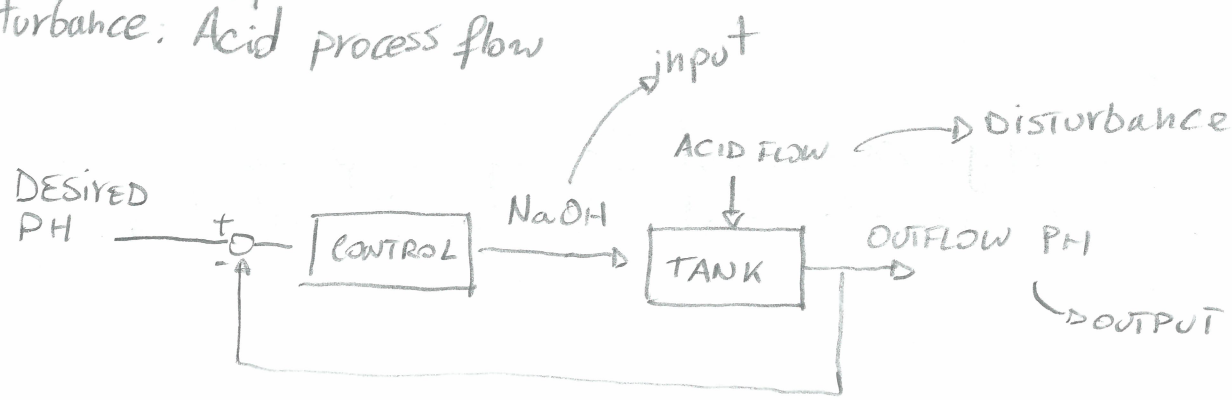
If $u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$ then $U(s) = 1/s$. The final value theorem tells us that

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sT(s)U(s) = \lim_{s\to 0} sT(s)\frac{1}{s} = \lim_{s\to 0} T(s) = T(0).$$

also called STATIC GAIN

EX 2.11

- input: NaOH solution
- output: The outflow PH
- disturbance: Acid process flow



EX 2.10

NB: all axes have equal origin and scaling! First thing we can understand is that the static gain $|G(0)|$ is either 1 or 2 ($\frac{1}{2}$ cannot be paired)

G_6 is also unstable, so it can't be.

Because of scaling, steps A/C are either G_1 or G_4 (because the gain is 1).

In step A we can't see the effect of the imaginary poles, so it should be a quick (fast) system. The slowest system between G_1 & G_4 is G_1 (check the pole with smallest real part), thus step A is G_4. It follows that step C is G_1.

In step B we clearly see OVERSHOOT. This can be caused by a ZERO.
⇒ It's G_3 (also it's very rapid in the beginning, which suggest the presence of a zero).
Step D then is G_5.

EX 2.5

- Let's consider B. The transfer function is $\frac{1}{s}$, thus the step response is $\frac{1}{s^2} \Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t \Rightarrow$ it's plot F.

- F: we have a pair of complex poles with negative real part. it will oscillate and converge to a static constant output ⇒ it's plot D.
- D: it's like B, but the negative real part makes it slower in the beginning ⇒ it's plot C.

- C: the effect of the poles should dissipate quickly, and the system output should converge to a constant output quickly. Can be either plot A or E. E is faster in the beginning, which suggest the presence of a zero ⇒ thus is plot A.

- A: the zero in the origin cancels at the imput step. We are left with a couple of negative real poles that will make the output converge to 0 ⇒ it's plot B.

- Finally, E is paired with plot E (notice the quick response and the "small" overshoot due to the zero).